Inequality involving triangles

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In a triangle ABC, with usual notions $p = (a+b+c)/2$ and inradius r, prove that

$$\sum \sqrt{\frac{ab(p-c)}{p}} \geq 6r.$$

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Noting that $\sqrt{\frac{(p-b)(p-c)}{bc}} = \sqrt{\frac{(p-a)(p-b)(p-c)}{p}} \cdot \sqrt{\frac{p}{bc(p-a)}} =$

$r\sqrt{\frac{p}{bc(p-a)}}$ we obtain that $\sum \sqrt{\frac{bc(p-a)}{p}} \geq 6r \Leftrightarrow \sum \frac{1}{r}\sqrt{\frac{bc(p-a)}{p}} \geq 6 \Leftrightarrow$

(**1**) $\quad \sum \sqrt{\frac{bc}{(p-b)(p-c)}} \geq 6.$ Since by Cauchy Inequality

$\sum \sqrt{\frac{(p-b)(p-c)}{bc}} \cdot \sum \sqrt{\frac{bc}{(p-b)(p-c)}} \geq 9$ remains to prove inequality

$9 \geq 6\sum \sqrt{\frac{(p-b)(p-c)}{bc}} \Leftrightarrow \sum \sqrt{\frac{(p-b)(p-c)}{bc}} \leq \frac{3}{2}.$

By AM-GM Inequality we have $\sum \sqrt{\frac{(p-b)(p-c)}{bc}} \leq \sum \frac{1}{2}\left(\frac{p-b}{c} + \frac{p-c}{b}\right) =$

$\sum \frac{1}{2}\left(\frac{p-b}{c} + \frac{p-a}{c}\right) = \frac{1}{2}\sum \frac{2p-a-b}{c} = \frac{3}{2}.$